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NONLINEAR ANISOTROPIC ELLIPTIC AND PARABOLIC EQUATIONS IN $\mathbb{R}^N$ WITH ADVECTION AND LOWER ORDER TERMS AND LOCALLY INTEGRABLE DATA

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Abstract. We prove existence and regularity results for distributional solutions in $\mathbb{R}^N$ for nonlinear elliptic and parabolic equations with general anisotropic diffusivities as well as advection and lower-order terms that satisfy appropriate growth conditions. The data are assumed to be merely locally integrable.

1. Introduction

In this paper we prove existence and regularity of distributional solutions in an appropriate function space for nonlinear anisotropic elliptic equations. A prototype example is

$$-\sum_{\ell=1}^{N} \frac{\partial}{\partial x_l}\left(\beta_l(x)\left|\frac{\partial u}{\partial x_l}\right|^{p_l-2}\frac{\partial u}{\partial x_l}\right) - \operatorname{div} g(u) + |u|^{s-1}u = f \quad \text{in } \mathbb{R}^N, \qquad N \geq 2, \tag{1.1}$$

where each $\beta_l : \mathbb{R}^N \to \mathbb{R}$ is a strictly positive and bounded function; $g = (g_1, \dots, g_N)$ is a continuous vector field with components that grow like $|u|^{s-\eta}$ for $s > 1$ and some $\eta \in (1, s)$; and f is locally integrable. We also prove corresponding results for nonlinear anisotropic parabolic equations. For (1.1) we assume that the exponents $p_1, \dots, p_N$ and s are restricted as follows:

$$\begin{cases} \overline{p} < N, \qquad \dfrac{1}{\overline{p}} = \dfrac{1}{N}\displaystyle\sum_{l=1}^{N}\dfrac{1}{p_l}, \\ p_l > 1 \quad \text{and} \quad p_l > \dfrac{\overline{p}(N-1)}{N(\overline{p}-1)}, \qquad l = 1, \dots, N, \\ s > p_l, \qquad l = 1, \dots, N. \end{cases} \tag{1.2}$$

We recall that for isotropic elliptic equations with $p_l = 2$ for $l = 1, \dots, N$ and $s > 1$, and no advection field, existence and uniqueness results for distributional solutions are proved in [7]. In the isotropic case with $p_l = p > 2 - \frac{1}{N}$ for $l = 1, \dots, N$ and $s > p - 1$, still with no advection field, existence and regularity results for distributional solutions are proved in [5]. The corresponding results for isotropic parabolic equations are developed in [6].

Compared to [5, 6], the main feature of of the present paper is the combination of an anisotropic diffusion operator, nonlinear advection and lower-order terms, a locally integrable right-hand side f, and an unbounded domain. In the case of the Dirichlet problem on a bounded domain, existence and regularity results for distributional solutions with L^1-data have been obtained in [4, 11] for a class of anisotropic elliptic and parabolic equations. For an anisotropic parabolic reaction-diffusion-advection system with a zero-flux boundary condition, still on a bounded domain, similar results are established in [2].

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Our main purpose is to prove the existence of at least one function $u \in L^s_{\mathrm{loc}}(\mathbb{R}^N)$ that possesses the regularity

$$u \in \bigcap_{l=1}^{N} W^{1,q_l}_{\mathrm{loc}}(\mathbb{R}^N), \qquad 1 \le \; q_l < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_l, \tag{1.3}$$

where $\overline{p}$ is defined in (1.2), and solves (1.1) in the distributional sense. The anisotropic Sobolev spaces appearing in (1.3) are defined in the next section. Observe that (1.2) implies $\overline{p} > 2 - \frac{1}{N}$ and thus $\frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_l > 1$, which is in accordance with the "isotropic" theory [5]. On the other hand, the condition $s > p_l$ in (1.2) is stronger than in [5]. This is a consequence of the anisotropic Sobolev inequality [17] that we have at our disposal here.

As in [5, 6], the strategy of an existence proof consist of deriving "good" a priori estimates for suitable approximate solutions $(u_\varepsilon)_{0<\varepsilon<1}$ (to which the standard variational framework applies) and passing to the limit as $\varepsilon \to 0$. There are two difficulties associated with this strategy. In view of the assumption that f is only locally integrable on $\mathbb{R}^N$, the first difficulty is to obtain suitable local a priori estimates on u_ε and the partial derivatives $\frac{\partial u_\varepsilon}{\partial x_l}$, $l = 1, \dots, N$, that are independent of ε. The second difficulty lies in passing to the limit in the nonlinear vector field $A(x, \nabla u_\varepsilon) + g(u_\varepsilon)$ and the nonlinear term $|u_\varepsilon|^{s-1} u_\varepsilon$.

The remaining part of the paper is organized as follows: In Section 2 we recall some basic notations and a Sobolev inequality for anisotropic Sobolev spaces. In addition, we prove an "interpolation" lemma that will be used later to obtain local a priori estimates. Our main "elliptic" results are stated in Section 3, while the proofs are given in Section 4. In Section 5 we briefly discuss the Dirichlet problem on a bounded domain. Finally, we convert our "elliptic" results to "parabolic" results in Section 6.

2. Anisotropic Sobolev spaces and a technical lemma

We start by recalling the notion of anisotropic Sobolev spaces. These spaces were introduced and studied by Nikolskii [14], Slobodeckii [16], and Troisi [17], and later by Trudinger [18] in the framework of Orlicz spaces.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipchitz boundary $\partial\Omega$. Let $p_1, \dots, p_N$ be N real numbers with $p_l \ge 1$, $l = 1, \dots, N$. With a slight abuse of the notation, we introduce the anisotropic Sobolev space

$$W^{1,p_l}(\Omega) = \left\{ u \in L^{p_l}(\Omega) : \frac{\partial u}{\partial x_l} \in L^{p_l}(\Omega) \right\},$$

which is a Banach space under the norm

$$\|u\|_{W^{1,p_l}(\Omega)} = \|u\|_{L^{p_l}(\Omega)} + \left\| \frac{\partial u}{\partial x_l} \right\|_{L^{p_l}(\Omega)},$$

for $l = 1, \dots, N$.

We recall the anisotropic Sobolev imbedding theorem due to Troisi [17] (see also [1]).

Theorem 2.1. *Suppose $u \in \bigcap_{l=1}^{N} W^{1,p_l}(\Omega)$, and set*

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{p_l}, \qquad r = \begin{cases} \overline{p}^\star := \frac{N\overline{p}}{N-\overline{p}}, & \text{if } \overline{p}^\star < N, \\ \text{any number from } [1,\infty), & \text{if } \overline{p}^\star \ge N. \end{cases}$$

Then there exists a constant C, depending on N, $p_1, \dots, p_N$ if $\overline{p} < N$ and also on r and meas(Ω) *if $\overline{p} \ge N$, such that*

$$\|u\|_{L^r(\Omega)} \le C \prod_{l=1}^{N} \left[\left\| \frac{\partial u}{\partial x_l} \right\|_{L^{p_l}(\Omega)} + \|u\|_{L^{p_l}(\Omega)} \right]^{\frac{1}{N}}. \tag{2.1}$$

Theorem 2.1 is used to prove the "interpolation" lemma below, which is a technical result we will use later to obtain a priori estimates. A similar result is found in [4] with $W^{1,p_l}(\Omega)$ replaced by $W_0^{1,p_l}(\Omega)$ in the case of a Dirichlet boundary condition.

Lemma 2.2. *Let $(u_\varepsilon)_{0<\varepsilon\leq 1}$ be a sequence in $\bigcap_{l=1}^{N} W^{1,p_l}(\Omega)$ with $\overline{p} \leq N$. Suppose that there exists a constant c, independent of ε, such that*

$$\|u_\varepsilon\|_{L^{p_l}(\Omega)} \leq c, \qquad l = 1, \dots, N, \tag{2.2}$$

and

$$\sup_{\gamma>0} \sum_{l=1}^{N} \int_{B_\gamma} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l} dx \leq c, \tag{2.3}$$

where $B_\gamma = \{x \in \Omega : \gamma \leq |u_\varepsilon| \leq \gamma + 1\}$ for $\gamma > 0$, or

$$\sum_{l=1}^{N} \int_\Omega \frac{\left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l}}{(1 + |u_\varepsilon|)^\gamma} dx \leq c. \tag{2.4}$$

Then for every q_l such that

$$1 \leq q_l < \frac{N(\overline{p} - 1)}{\overline{p}(N - 1)} p_l, \tag{2.5}$$

there exists a constant C, depending on Ω, N, $p_1, \dots, p_N$, $q_1, \dots, q_N$, and c, but not ε, such that

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l}(\Omega)} \leq C, \qquad l = 1, \dots, N, \tag{2.6}$$

and

$$\|u_\varepsilon\|_{L^{\overline{q}}(\Omega)} \leq C, \qquad \frac{1}{\overline{q}} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{q_l}. \tag{2.7}$$

Proof. We adapt the proof in [3, 4] to our setting. Let $q_l < p_l$ and $\gamma_0 \geq 1$. Then, using (2.3),

$$\begin{aligned} \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_l} dx &= \sum_{\gamma=0}^{\gamma_0-1} \int_{B_\gamma} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_l} dx + \sum_{\gamma=\gamma_0}^{\infty} \int_{B_\gamma} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_l} dx \\ &\leq C_{\gamma_0} + \sum_{\gamma=\gamma_0}^{\infty} \int_{B_\gamma} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_l} dx \\ &\leq C_{\gamma_0} + \sum_{\gamma=\gamma_0}^{\infty} \left(\int_{B_\gamma} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l} dx \right)^{\frac{q_l}{p_l}} (\operatorname{meas}(B_\gamma))^{1-\frac{q_l}{p_l}} \\ &\leq C_{\gamma_0} + C_1 \sum_{\gamma=\gamma_0}^{\infty} (\operatorname{meas}(B_\gamma))^{\frac{p_l-q_l}{p_l}}. \end{aligned} \tag{2.8}$$

Clearly, $\frac{1}{\gamma^{\overline{q}^\star}} \int_{B_\gamma} |u_\varepsilon|^{\overline{q}^\star} dx \geq \operatorname{meas}(B_\gamma)$. From this estimate and Hölder's inequality, we deduce

$$\begin{aligned} \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_l} dx &\leq C_2 + C_3 \sum_{\gamma=\gamma_0}^{\infty} \frac{1}{\gamma^{\frac{p_l-q_l}{p_l}\overline{q}^\star}} \left(\int_{B_\gamma} |u_\varepsilon|^{\overline{q}^\star} dx \right)^{\frac{p_l-q_l}{p_l}} \\ &\leq C_2 + C_4 \left(\sum_{\gamma=\gamma_0}^{\infty} \frac{1}{\gamma^{\frac{p_l-q_l}{q_l}\overline{q}^\star}} \right)^{\frac{q_l}{p_l}} \left(\sum_{\gamma=\gamma_0}^{\infty} \int_{B_\gamma} |u_\varepsilon|^{\overline{q}^\star} dx \right)^{\frac{p_l-q_l}{p_l}}. \end{aligned} \tag{2.9}$$

The anisotropic Sobolev inequality (2.1) gives

$$\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}^\star}} \le C_5 \prod_{l=1}^N \left[\left(\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{q_l} dx\right)^{\frac{1}{q_l}} + \left(\int_\Omega |u_\varepsilon|^{q_l}\,dx\right)^{\frac{1}{q_l}}\right]^{\frac{1}{N}}, \tag{2.10}$$

where $\overline{q}^\star := \dfrac{N\overline{q}}{N-\overline{q}}$ (note $\overline{q} \in (1,N)$). Since $q_l < p_l$, it follows from (2.10) and (2.2) that

$$\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}^\star}} \le C_6 \prod_{l=1}^N \left(\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{q_l} dx\right)^{\frac{1}{q_l N}} + C_7. \tag{2.11}$$

By (2.11), (2.9), and the fact that $\dfrac{p_l - q_l}{q_l}\overline{q}^\star > 1$ thanks to (2.5),

$$\begin{aligned}\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}^\star}} &\le C_9 + C_{10} \prod_{l=1}^N \left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{p_l-q_l}{q_l p_l N}} \\ &= C_9 + C_{10}\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\sum_{l=1}^N \frac{p_l-q_l}{q_l p_l N}} \\ &= C_9 + C_{10}\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}}-\frac{1}{\overline{p}}}.\end{aligned}$$

In other words,

$$\|u_\varepsilon\|_{L^{\overline{q}^\star}(\Omega)} \le C_9 + C_{10}\,\|u_\varepsilon\|^a_{L^{\overline{q}^\star}(\Omega)}, \qquad a := \frac{\overline{p}-\overline{q}}{\overline{q}\,\overline{p}}\overline{q}^\star.$$

One checks easily that the assumption $\overline{p} < N$ implies $a < 1$, and we can therefore conclude that (2.7) holds. Moreover, (2.6) follows from (2.9) and (2.7).

Let $q_l = \kappa p_l$, $l = 1,\dots,N$, for any $\kappa \in \left(0, \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_l\right)$. Let $\lambda = \frac{1-\kappa}{\kappa}\overline{q}^\star$, so that $\gamma\frac{q_l}{p_l-q_l} = \overline{q}^\star$. Recalling $\frac{p_l-q_l}{q_l}\overline{q}^\star > 1$, we see that $\lambda > 1$. Using Hölder's inequality and then estimate (2.4), we obtain

$$\begin{aligned}\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{q_l} dx &\le \left(\int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)^\gamma}\,dx\right)^{\frac{q_l}{p_l}} \left(\int_\Omega (1+|u_\varepsilon|)^{\gamma\frac{q_l}{p_l-q_l}}\,dx\right)^{\frac{p_l-q_l}{p_l}} \\ &\le C_{11}\left(\int_\Omega (1+|u_\varepsilon|)^{\gamma\frac{q_l}{p_l-q_l}}\,dx\right)^{\frac{p_l-q_l}{p_l}} \le C_{12}\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{p_l-q_l}{p_l}} + C_{13}.\end{aligned} \tag{2.12}$$

Inserting this into (2.11) and proceeding as above, we conclude that (2.6) and (2.7) hold under condition (2.4) instead of (2.3). □

3. Statements of results

Instead of (1.1) we will consider more general nonlinear anisotropic elliptic equations of the form

$$-\mathrm{div}A(x,\nabla u) - \mathrm{div}g(x,u) + h(x,u) = f(x) \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

The vector field $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ has components $a_l : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $l = 1,\dots,N$, and we assume that there exist two constants C_A and C'_A such that for all $\xi_1, \xi_2 \in \mathbb{R}^N$ and for a.e. x

$$A(x,\xi)\cdot\xi \ge C_A \sum_{l=1}^N |\xi|^{p_l}, \tag{3.2}$$

$$|a_l(x,\xi)| \le C'_A\left(1 + \sum_{\ell=1}^N |\xi_\ell|^{p_\ell - 1}\right), \qquad l = 1,\dots,N, \tag{3.3}$$

$$[A(x,\xi_1) - A(x,\xi_2)]\,[\xi_1 - \xi_2] > 0, \qquad \xi_1 \ne \xi_2. \tag{3.4}$$

The advection field $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ has continuous components $g_l : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $l = 1, \ldots, N$, and satisfies the following conditions:

$$|g(x,\sigma)| \le C_g |\sigma|^{s-\eta}, \qquad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } \sigma \in \mathbb{R}. \tag{3.5}$$

$$|\mathrm{div}_x g(x,\sigma)| \le C_g' |\sigma|^{s-\eta}, \qquad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } \sigma \in \mathbb{R}, \tag{3.6}$$

for some constants C_g, C_g' and some $\eta \in (1, s)$.

The nonlinear function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be measurable in $x \in \mathbb{R}^N$ for all $\sigma \in \mathbb{R}$ and continuous in $\sigma \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^N$. Furthermore,

$$h(x,\sigma)\sigma \ge 0, \qquad \text{for all } \sigma \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N, \tag{3.7}$$

$$\sup\{|h(x,\sigma)| : |\sigma| \le \tau\} \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \qquad \forall \tau \in \mathbb{R}. \tag{3.8}$$

Finally, there should exist $s > p_l$, $l = 1, \ldots, N$, such that

$$h(x,\sigma)\mathrm{sign}(\sigma) \ge |\sigma|^s, \qquad \text{for all } \sigma \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \tag{3.9}$$

We look for distributional solutions to (3.1) in the following sense:

Definition 3.1. A distributional solution of (3.1) is a function $u : \mathbb{R}^N \to \mathbb{R}$ such

$$u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^s_{\mathrm{loc}}(\mathbb{R}^N), \qquad A(x, \nabla u) \in \left(L^1_{\mathrm{loc}}(\mathbb{R}^N)\right)^N,$$

and

$$\int_{\mathbb{R}^N} (A(x,\nabla u) + g(x,u)) \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^N} h(x,u)\varphi \, dx = \int_{\mathbb{R}^N} f\varphi \, dx, \qquad \forall \varphi \in C^1_c(\mathbb{R}^N). \tag{3.10}$$

Note that (1.3) and the conditions on g, h imply that all the terms in (3.10) are well-defined.

Our main results are collected in the following theorem:

Theorem 3.1. *Assume* (3.2)-(3.9) *hold and that the corresponding exponents $p_1, \ldots, p_N$ and s are restricted as in* (1.2). *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then* (3.1) *has at least one distributional solution u. If $f \ge 0$, then $u \ge 0$. Moreover, u possesses the regularity stated in* (1.3). *Finally, if $f \in L^1(\mathbb{R}^N)$ and $\overline{p} > N$, then $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$.*

4. Proof of Theorem 3.1

For any $R > 0$, let $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. In what follows, it is always understood that ε takes values in a sequence tending to zero. Let $(f_\varepsilon)_{0<\varepsilon<1} \subset C^\infty_c(\Omega)$ be a sequence of smooth approximations of f such that

$$\begin{cases} |f_\varepsilon| \le \dfrac{1}{\varepsilon} \quad \text{and} \quad |f_\varepsilon| \le |f|; \\ f_\varepsilon \to f \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0. \end{cases} \tag{4.1}$$

Then classical results, see, e.g., [13, 12, 10], provide us with the existence of a sequence of functions

$$(u_\varepsilon)_{0<\varepsilon\le 1} \subset \bigcap_{l=1}^N W_0^{1,p_l}\left(B_{\frac{1}{\varepsilon}}\right) \cap L^s(B_{\frac{1}{\varepsilon}}),$$

each of them satisfying the weak formulation

$$\int_{B_{\frac{1}{\varepsilon}}} (A(x,\nabla u_\varepsilon) + g(x,u_\varepsilon)) \cdot \nabla\varphi \, dx + \int_{B_{\frac{1}{\varepsilon}}} h(x,u_\varepsilon)\varphi = \int_{B_{\frac{1}{\varepsilon}}} f_\varepsilon \varphi \, dx, \tag{4.2}$$

for all $\varphi \in \bigcap_{l=1}^N W_0^{1,p_l}\left(B_{\frac{1}{\varepsilon}}\right) \cap L^\infty\left(B_{\frac{1}{\varepsilon}}\right)$, where

$$W_0^{1,p_l}\left(B_{\frac{1}{\varepsilon}}\right) = \left\{u \in W_0^{1,1}\left(B_{\frac{1}{\varepsilon}}\right) : \frac{\partial u}{\partial x_l} \in L^{p_l}\left(B_{\frac{1}{\varepsilon}}\right)\right\}.$$

The proof of Theorem 3.1 consists of three main steps. First, we prove ε-uniform local a priori estimates for u_ε, which imply a.e. convergence of u_ε. Second, we prove strong L^1_{loc} convergence of the nonlinear terms in (4.2). Finally, we complete the proof of Theorem 3.1 by passing to the limit in (4.1) as $\varepsilon \to 0$.

In the remaining part of this paper, we use C, C_1, C_2, etc. to denote constants that are independent of ε.

4.1. A priori estimates.

Proposition 4.1. *Assume* (3.2)-(3.9) *hold, and that the exponents* $p_1, \dots, p_N$ *and* s *are restricted as in* (1.2). *Set* $R := \frac{1}{\varepsilon}$, *and let* ρ *be any number such that* $0 < 2\rho < R$. *Then, there exist a constant* C, *not depending on* ε, *such that*

$$\|u_\varepsilon\|_{L^s(B_\rho)} \le C \tag{4.3}$$

and

$$\|h(x,u_\varepsilon)\|_{L^1(B_\rho)} \le C. \tag{4.4}$$

Moreover, for every $1 \le q_l < \dfrac{N(\overline{p}-1)}{\overline{p}(N-1)} p_l$ *there exists a constant* C, *depending on* B_ρ, N, $p_1, \dots, p_N$, $q_1, \dots, q_N$, $\|f\|_{L^1(B_{2\rho})}$ *but not* ε, *such that*

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l}(B_\rho)} \le C, \qquad l = 1, \dots, N, \tag{4.5}$$

and

$$\|u_\varepsilon\|_{L^{\overline{q}}(B_\rho)} \le C, \qquad \frac{1}{\overline{q}} := \frac{1}{N} \sum_{l=1}^N \frac{1}{q_l}. \tag{4.6}$$

Proof. Following [5], we introduce for $\gamma > 1$ the test function

$$\varphi_\gamma(\sigma) = \begin{cases} (\gamma-1)\displaystyle\int_0^\sigma \frac{1}{(1+t)^\gamma}\,dt = 1 - \frac{1}{(1+\sigma)^{\gamma-1}}, & \sigma \ge 0, \\ -\varphi_\gamma(-\sigma), & \sigma < 0, \end{cases} \tag{4.7}$$

and a smooth cut-off function $\theta = \theta(x)$ that is supported in the ball $B_{2\rho}$ (recall $0 < 2\rho < R$) such that $0 \le \theta \le 1$, $\theta(x) = 1$ for $|x| \le \rho$, and $|\nabla\theta| \le 2/\rho$. Observe that $|\varphi_\gamma| \le 1$ and, by assuming $\rho \ge 2$, there holds $|\nabla\theta| \le 1$.

Let $\alpha > 1$. Inserting $\varphi = \varphi_\gamma(u_\varepsilon)\theta^\alpha$ into (4.2) gives

$$\begin{aligned} &\int_{B_R} A(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon \varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx + \int_{B_R} g(x,u_\varepsilon)\cdot\nabla u_\varepsilon\varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx \\ &\quad + \int_{B_R} h(x,u_\varepsilon)\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx + \alpha\int_{B_R} A(x,\nabla u_\varepsilon)\cdot\nabla\theta\,\varphi_\gamma(u_\varepsilon)\theta^{\alpha-1}\,dx \\ &\quad\quad + \alpha\int_{B_R} g(x,u_\varepsilon)\cdot\nabla\theta\varphi_\gamma(u_\varepsilon)\theta^{\alpha-1}\,dx = \int_{B_R} f\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx. \end{aligned} \tag{4.8}$$

Now we choose γ and α so that (recall from (3.5) and (3.6) that $\eta \in (1,s)$)

$$1 < \gamma < \frac{s}{p_l - 1}, \qquad \alpha > \max\left\{s, \frac{s}{\eta-1}\right\} \quad \text{and} \quad \alpha > \frac{p_l s}{s - \gamma(p_l-1)}, \qquad l = 1, \dots, N. \tag{4.9}$$

Let us introduce the vector field $G = (G_1, ..., G_N)$ defined by

$$G_l(x,\sigma) = \int_0^\sigma g_l(x,t)\varphi_\gamma'(t)\,dt, \qquad l = 1, \dots, N.$$

Using the divergence theorem, $G(0)=0$, (3.5) and (3.6), the condition (4.9) on α, $|\nabla\theta|\le 1$, $\theta^\alpha\le\theta^{\alpha-1}$, and Young's inequality, we estimate as follows:

$$
\begin{aligned}
&\left|\int_{B_R} g(x,u_\varepsilon)\cdot\nabla u_\varepsilon\varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx\right| \\
&\quad=\left|\int_{B_R}\operatorname{div}G(u_\varepsilon)\theta^\alpha\,dx-\int_{B_R}\left(\int_0^{u_\varepsilon}\operatorname{div}_x g_l(x,t)\varphi_\gamma'(t)\,dt\right)\theta^\alpha\,dx\right| \\
&\quad=\left|-\int_{B_R}\alpha\theta^{\alpha-1}G(u_\varepsilon)\cdot\nabla\theta\,dx-\int_{B_R}\left(\theta^\alpha\int_0^{u_\varepsilon}\operatorname{div}_x g(x,t)\varphi_\gamma'(t)\,dt\right)dx\right| \\
&\quad\le C_1\int_{B_R}|u_\varepsilon|^{s-\eta+1}\theta^{\alpha-1}\,dx+C_2\int_{B_R}|u_\varepsilon|^{s-\eta+1}\theta^\alpha\,dx \\
&\quad\le C_3\int_{B_R}|u_\varepsilon|^{s-\eta+1}\theta^{\alpha-1}\,dx=C_3\int_{B_R}|u_\varepsilon|^{s-\eta+1}\theta^{\frac{s-\eta+1}{s}\alpha}\theta^{\frac{\eta-1}{s}\alpha-1}\,dx \\
&\quad\le\frac18\int_{B_R}\varphi_\gamma(1)\,|u_\varepsilon|^s\,\theta^\alpha\,dx+C_4\int_{B_R}\theta^{\alpha-\frac{s}{\eta-1}}\,dx \\
&\quad\le\frac18\int_{B_R}\varphi_\gamma(1)\,|u_\varepsilon|^s\,\theta^\alpha\,dx+C_5\operatorname{meas}(B_{2\rho})\,.
\end{aligned}
\tag{4.10}
$$

Similarly, we deduce the estimate

$$
\begin{aligned}
&\left|\int_{B_R} g(x,u_\varepsilon)\cdot\nabla\theta\varphi_\gamma(u_\varepsilon)\theta^{\alpha-1}\,dx\right| \\
&\quad\le\int_{B_\rho}|g(x,u_\varepsilon)|\,\theta^{\alpha-1}\,dx\le\frac18\int_{B_R}\varphi_\gamma(1)\,|u_\varepsilon|^s\,\theta^\alpha\,dx+C_6\operatorname{meas}(B_{2\rho})\,.
\end{aligned}
\tag{4.11}
$$

Using the structure conditions (3.2) and (3.3) in (4.8) along with (4.10) and (4.11), we get

$$
\begin{aligned}
&C_A\int_{B_R}\sum_{l=1}^N\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}\varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx+\int_{B_R}h(x,u_\varepsilon)\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx \\
&\quad\le\int_{B_{2\rho}}|f|+C_7\int_{B_R}\sum_{l=1}^N\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l-1}\theta^{\alpha-1}\,dx \\
&\qquad+\frac14\int_{B_R}\varphi_\gamma(1)\,|u_\varepsilon|^s\,\theta^\alpha\,dx+C_8\operatorname{meas}(B_{2\rho})\,.
\end{aligned}
\tag{4.12}
$$

An application of Young's inequality gives

$$
\begin{aligned}
\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l-1}\theta^{\alpha-1}&=\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l-1}\left(\varphi_\gamma'(u_\varepsilon)\right)^{\frac{p_l-1}{p_l}}\theta^{\alpha\frac{p_l-1}{p_l}}\left(\varphi_\gamma'(u_\varepsilon)\right)^{\frac{1-p_l}{p_l}}\theta^{\alpha\frac{\alpha-p_l}{p_l}} \\
&\le\frac{C_A}{2C_7}\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}\varphi_\gamma'(u_\varepsilon)\theta^\alpha+C_9\frac{\theta^{\alpha-p_l}}{\varphi_\gamma'(u_\varepsilon)^{p_l-1}} \\
&=\frac{C_A}{2C_7}\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p}\varphi_\gamma'(u_\varepsilon)\theta^\alpha+C_{10}\left(1+|u_\varepsilon|\right)^{\gamma(p_l-1)}\theta^{\alpha-p_l} \\
&=\frac{C_A}{2C_7}\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p}\varphi_\gamma'(u_\varepsilon)\theta^\alpha+C_{11}\,|u_\varepsilon|^{\gamma(p_l-1)}\,\theta^{\alpha-p_l}+C_{12}\theta^{\alpha-p_l}.
\end{aligned}
\tag{4.13}
$$

We can estimate the last term in (4.13) by another application of Young's inequality and (3.9):

$$
\begin{aligned}
C_{11}\,|u_\varepsilon|^{\gamma(p_l-1)}\,\theta^{\alpha-p_l}&=C_{11}\,|u_\varepsilon|^{\gamma(p_l-1)}\,\theta^{\alpha\frac{\gamma(p_l-1)}{s}}\theta^{\alpha\frac{s-\gamma(p_l-1)}{s}-p_l} \\
&\le\frac{\varphi_\gamma(1)}{4}\,|u_\varepsilon|^s\,\theta^\alpha+C_{13}\theta^{\alpha-\frac{p_ls}{s-\gamma(p_l-1)}}.
\end{aligned}
\tag{4.14}
$$

From (4.7) and (3.9), it follows that

$$
h(x,\sigma)\varphi_\gamma(\sigma)\ge|\sigma|^s\,\varphi_\gamma(1),\qquad\text{for }\sigma\ge1\text{ and a.e. }x\in\mathbb{R}^N,
$$

and hence

$$|\sigma|^s \le h(x,\sigma)\frac{\varphi_\gamma(\sigma)}{\varphi_\gamma(1)} + 1, \qquad \text{for } \sigma \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \tag{4.15}$$

Using (4.13), (4.14), and (4.15) in (4.12) we obtain

$$\begin{aligned} &\frac{C_A}{2}\int_{B_R}\sum_{l=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l} \varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx + \frac{1}{2}\int_{B_R} h(x,u_\varepsilon)\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx \\ &\quad \le \int_{B_{2\rho}} |f|\,dx + C_{14}\mathrm{meas}\,(B_{2\rho})\,. \end{aligned} \tag{4.16}$$

Using the definitions of φ_γ and θ, we obtain from (4.16) and (4.15) that

$$\int_{B_\rho} |u_\varepsilon|^s\,dx \le C_{15}, \tag{4.17}$$

which proves (4.3) and, via (3.9), also (4.4). Moreover, it follows that

$$\sum_{l=1}^N \int_{B_\rho} \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)^\gamma}\,dx \le C_{16}. \tag{4.18}$$

Estimates (4.5) and (4.6) are direct consequences of (4.17), (4.18), and Lemma 2.2.

□

4.2. **Strong convergence.** Given any $\rho > 0$, let ε be such that $\frac{1}{\varepsilon} > 2\rho$. In view of Proposition 4.1, u_ε is uniformly (in ε) bounded in $W^{1,q_0}(B_\rho)$, where

$$q_0 := \min_{1\le l\le N} q_l, \tag{4.19}$$

and $q_1,\dots,q_N$ are restricted as in Proposition 4.1. Without loss of generality, we can therefore assume that

$$\begin{cases} u_\varepsilon \to u \quad \text{strongly in } L^{q_0}(B_\rho) \text{ and a.e. in } B_\rho, \\ h(x,u_\varepsilon) \to h(x,u),\ g(x,u_\varepsilon) \to g(x,u) \quad \text{a.e. in } B_\rho. \end{cases} \tag{4.20}$$

By a standard diagonal process, we can in fact assume that $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$, $u_\varepsilon \to u$ weakly in $W^{1,q_0}_{\mathrm{loc}}(\mathbb{R}^N)$, and $h(x,u_\varepsilon) \to h(x,u)$, $g(x,u_\varepsilon) \to g(x,u)$ a.e. in $\mathbb{R}^N$.

For passing to the limit in (4.2), we prove first the convergence in $L^1(B_\rho)$ of the sequences $(h(x,u_\varepsilon))_{0<\varepsilon\le 1}$, $(g(x,u_\varepsilon))_{0<\varepsilon\le 1}$ to respectively $h(x,u)$, $g(x,u)$.

Proposition 4.2. *Assume* (3.2)-(3.9) *hold, and that the corresponding exponents* $p_1,\dots,p_N$ *and* s *are restricted as in* (1.2). *Then the sequences* $(h(x,u_\varepsilon))_{0<\varepsilon\le 1}$ *and* $(g(x,u_\varepsilon))_{0<\varepsilon\le 1}$ *converge to respectively* $h(x,u)$ *and* $g(x,u)$ *a.e. in* $\mathbb{R}^N$ *and strongly in* $L^1(B_\rho)$ *for any* $\rho > 0$.

Proof. In view of (4.20) and a theorem of Vitali (see, e.g., [8]), it is sufficient to establish the equi-integrability of $(h(x,u_\varepsilon))_{0<\varepsilon\le 1}$ on B_ρ. To this end, we follow [5, 6] and introduce for $\gamma,\beta > 1$ the test function $\varphi_{\gamma,\beta}$ defined by

$$\varphi_{\gamma,\beta}(\sigma) = \begin{cases} \varphi_\gamma(\sigma-\beta), & \sigma \ge \beta \\ 0, & |\sigma| < \beta \\ -\varphi_{\gamma,\beta}(-\sigma), & \sigma \le -\beta, \end{cases} \tag{4.21}$$

where φ_γ is defined in (4.7). Let $\alpha > 1$. Inserting $\varphi = \varphi_{\gamma,\beta}(u_\varepsilon)\theta^\alpha$ into (4.2) and proceeding more or less as we did up to (4.16), we find

$$\begin{aligned} &\frac{C_A}{2}\int_{B_R}\sum_{l=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l} \varphi_{\gamma,\beta}'(u_\varepsilon)\theta^\alpha\,dx + \frac{1}{2}\int_{B_R} h(x,u_\varepsilon)\varphi_{\gamma,\beta}(u_\varepsilon)\theta^\alpha\,dx \\ &\quad \le \int_{B_{2\rho}\cap\{|u_\varepsilon|\ge\beta\}} |f|\,dx + C_1\mathrm{meas}\,(B_{2\rho}\cap\{|u_\varepsilon|\ge\beta\})\,. \end{aligned} \tag{4.22}$$

Since $f \in L^1(B_{2\rho})$ and u_ε is bounded in $L^1(B_{2\rho})$ uniformly with respect to ε,

$$\int_{B_{2\rho}\cap\{|u_\varepsilon|\geq\beta\}} |f|\,dx + \mathrm{meas}(B_{2\rho}\cap\{|u_\varepsilon|\geq\beta\}) \to 0, \qquad \text{as } \beta\to\infty. \tag{4.23}$$

From (4.21), (3.7), (4.22), and (4.23), we conclude that

$$\int_{B_\rho\cap\{|u_\varepsilon|\geq\beta+1\}} |h(x,u_\varepsilon)|\,dx \leq C\int_{B_R} h(x,u_\varepsilon)\varphi_{\gamma,\beta}(u_\varepsilon)\theta^\alpha\,dx \stackrel{\beta\to\infty}{\longrightarrow} 0 \quad (\text{uniformly in } \varepsilon).$$

By (3.8), this implies the desired equi-integrability of $(h(x,u_\varepsilon))_{0<\varepsilon\leq 1}$.

From (3.5) and the convergence proof just given, we deduce easily that $g(x,u_\varepsilon)$ converges to $g(x,u)$ a.e. in $\mathbb{R}^N$ and strongly in $L^1(B_\rho)$ for any $\rho>0$. □

Proposition 4.3. *Assume* (3.2)-(3.9) *hold, and that the corresponding exponents* $p_1,\dots,p_N$ *and* s *are restricted as in* (1.2). *Then the sequence* $(A(x,\nabla u_\varepsilon))_{0<\varepsilon\leq 1}$ *converges to* $A(x,\nabla u)$ *a.e. in* $\mathbb{R}^N$ *and strongly in* $L^1(B_\rho)$ *for any* $\rho>0$.

Proof. As in [5, 6], we prove first that the sequence $(\nabla u_\varepsilon)_{0<\varepsilon\leq 1}$ converges to ∇u in measure on B_ρ, which implies a.e. convergence after passing to a suitable subsequence. It suffices to show that $(\nabla u_\varepsilon)_{0<\varepsilon\leq 1}$ is a Cauchy sequence in measure on B_ρ, i.e., for any $\mu>0$,

$$\mathrm{meas}\left(\{x\in B_\rho : |(\nabla u_{\varepsilon'}-\nabla u_\varepsilon)(x)|\geq\mu\}\right)\to 0, \qquad \text{as } \varepsilon,\varepsilon'\to 0.$$

For any $\gamma,\delta>0$, we have

$$\{x\in B_\rho : |(\nabla u_{\varepsilon'}-\nabla u_\varepsilon)(x)|\geq\mu\}\subset L_1\cup L_2\cup L_3\cup L_4,$$

where $L_1=\{x\in B_\rho : |\nabla u_\varepsilon(x)|\geq\gamma\}$, $L_2=\{x\in B_\rho : |\nabla u_{\varepsilon'}(x)|\geq\gamma\}$,

$$L_3=\{x\in B_\rho : |(u_\varepsilon-u_{\varepsilon'})(x)|\geq\delta\},$$

and

$$L_4=\{x\in B_\rho : |(\nabla u_\varepsilon-\nabla u_{\varepsilon'})(x)|\geq\mu, |\nabla u_\varepsilon(x)|\leq\gamma, |\nabla u_{\varepsilon'}(x)|\leq\gamma, |(u_\varepsilon-u_{\varepsilon'})(x)|\leq\delta\}.$$

In view of Proposition 4.1, by choosing γ large we can make $\mathrm{meas}(L_1)$ and $\mathrm{meas}(L_2)$ arbitrarily small. Since $(u_\varepsilon)_{0<\varepsilon\leq 1}$ is a Cauchy sequence in $L^1(B_\rho)$, then, for $\delta>0$ fixed, $\mathrm{meas}(L_3)$ tends to 0 as $\varepsilon,\varepsilon'\to 0$. It remains to control $\mathrm{meas}(L_4)$. Since the set of (ξ_1,ξ_2) such that $|\xi_1|\leq\gamma$, $|\xi_2|\leq\gamma$, and $|\xi_1-\xi_2|\leq\mu$ is a compact set and $\xi\mapsto A(x,\xi)$ is continuous for a.e. $x\in B_\rho$, the quantity

$$[A(x,\xi_1)-A(x,\xi_2)]\,[\xi_1-\xi_2]$$

reaches its minimum value on this compact set, and we will denote it by $q(x)$. By (3.4), it is not hard to verify that $q(x)>0$ a.e. in B_ρ. Consequently, for any $\beta>0$ there exists $\beta'>0$ such that

$$\int_{L_4} q(x)\,dx<\beta' \Longrightarrow \mathrm{meas}(L_4)\leq\beta. \tag{4.24}$$

Hence, it is sufficient to show that for any given $\beta'>0$, one can produce a small enough $\delta>0$ such that

$$\int_{L_4} q(x)\,dx<\beta'. \tag{4.25}$$

For any $\delta>0$, define $T_\delta(z)=\min(\delta,\max(z,-\delta))$. Note that T_δ is a Lipschitz continuous function satisfying $0\leq|T_\delta(z)|\leq\delta$. By the definitions of $q(x)$ and L_4, we have

$$\begin{aligned}\int_{L_4} q(x)\,dx &\leq \int_{L_4}[A(x,\nabla u_\varepsilon)-A(x,\nabla u_{\varepsilon'})]\,[\nabla u_\varepsilon-\nabla u_{\varepsilon'}]\,\mathbf{1}_{\{|u_\varepsilon-u_{\varepsilon'}|\leq\delta\}}\,dx\\ &=\int_{L_4}[A(x,\nabla u_\varepsilon)-A(x,\nabla u_{\varepsilon'})]\,\nabla T_\delta\left(u_\varepsilon-u_{\varepsilon'}\right)\,dx.\end{aligned} \tag{4.26}$$

Let θ be the cut-off function used in the proof of Proposition 4.1. Set $p^0:=\max_{1\leq l\leq N} p_l$, and let q_0 be the number defined in (4.19). Thanks to Proposition 4.1, we can find a $q\in\left[p^0-1,q_0\right)$ such that $\left\|\frac{\partial u_\varepsilon}{\partial x_l}\right\|_{L^q(B_{2\rho})}$ is bounded independently of ε for all $l = 1, \dots, N$. Specifying $T_\delta(u_\varepsilon - u_{\varepsilon'})\,\theta$ as test function in the weak formulations for u_ε and $u_{\varepsilon'}$ and then subtracting the results, we find

$$\begin{aligned}
&\int_{B_\rho} [A(x,\nabla u_\varepsilon) - A(x,\nabla u_{\varepsilon'})] \cdot \nabla T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx \\
&\le 2\delta \left[C_1 + C_2 \int_{B_{2\rho}} \sum_{l=1}^{N} \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l - 1} dx + C_3 \|u_\varepsilon\|_{L^s(B_{2\rho})} + \|f\|_{L^1(B_{2\rho})} \right] \\
&\le 2\delta \left[C_1 + C_4 \int_{B_{2\rho}} \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{q} dx + C_3 \|u_\varepsilon\|_{L^s(B_{2\rho})} + \|f\|_{L^1(B_{2\rho})} \right] \\
&\overset{\delta\to 0}{\longrightarrow} 0 \quad \text{(uniformly in } \varepsilon \text{ and } \varepsilon').
\end{aligned} \tag{4.27}$$

For δ small enough, we have from (4.26) and (4.27) that (4.25) holds, and, by (4.24), also that $\operatorname{meas}(L_4) \le \beta$. Thus, we have the convergence of $(\nabla u_\varepsilon)_{0<\varepsilon\le 1}$ to ∇u in measure. Thanks to this measure convergence and (4.5), we can finally conclude that along a subsequence

$$A(x,\nabla u_\varepsilon) \to A(x,\nabla u) \quad \text{strongly in } L^1(B_\rho).$$

□

4.3. **Completing the proof of Theorem 3.1.** In view of the previous results, we can indeed send $\varepsilon \to 0$ in the weak formulation (4.2) with $\varphi \in C_c^1(\mathbb{R}^N)$, thereby obtaining the existence of a distributional solution (in the sense of Definition 3.1) to (3.1), which possesses the regularity stated in (1.3). If $f \ge 0$, then $u_\varepsilon \ge 0$ a.e. in $\mathbb{R}^N$ for any $\varepsilon > 0$. Hence the limit u is also nonnegative. The L^∞_{loc}-bound for u_ε is proved by replacing $\overline{q}^\star$ in the proof Lemma 2.2 by any number $r \in [1,\infty)$ and using Theorem 2.1.

5. The Dirichlet problem on a bounded domain

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \ge 2)$. In this section we wish to point out that the existence result obtained in the previous section also applies to the Dirichlet problem on a bounded domain. In fact, on a bounded domain (under stronger assumptions) it is possible to prove that the constructed distributional solution has regularity corresponding to the limiting case of equality in the upper bound on q_l in (1.3). Our results generalize those obtained in [4] to general problems of the form

$$\begin{cases} -\operatorname{div} A(x,\nabla u) - \operatorname{div} g(x,u) + h(x,u) = f(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

where A, g, h satisfy the conditions stated in (3.2)-(3.9).

Theorem 5.1. *Assume* (3.2)-(3.9) *hold, and that the exponents $p_1, \dots, p_N$ and s are restricted as in* (1.2)*. In addition, assume*

$$p_l > \frac{1}{1+\eta - s}, \quad l = 1,\dots,N, \qquad \eta \in (s-1, s), \tag{5.2}$$

where η is given in (3.5) *and* (3.6)*. Let $f \in L^1(\Omega)$. Then the exists at least one function $u \in W_0^{1,1}(\Omega) \cap L^s(\Omega)$ such that $A(x,\nabla u) \in L^1(\Omega)$, $\frac{\partial u}{\partial x_l} \in L^{q_l}(\Omega)$ with $1 \le q_l < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_l$, $l = 1,\dots,N$, and* (5.1) *holds in the distribution sense. If $f \in L^1 \log L^1(\Omega)$, i.e.,*

$$\int_\Omega |f| \log(1 + |f|)\,dx < \infty,$$

then there exists a distributional solution u of (5.1) *such that*

$$\frac{\partial u}{\partial x_l} \in L^{q_l}(\Omega), \quad q_l = \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_l, \quad l = 1,\dots,N. \tag{5.3}$$

Proof. Let $(u_\varepsilon)_{0<\varepsilon\leq 1}$ be a sequence of approximate solutions satisfying the weak formulation (4.2) with $B_{\frac{1}{\varepsilon}}$ replaced by Ω. The first part of the theorem can be proved by adapting the proof of Theorem 3.1. Let us prove (5.3). Since, by (5.2), $\frac{(s-\eta)p_l}{p_l-1} < 1$, we deduce from (3.5)

$$
\begin{aligned}
\int_\Omega \left| g(x,u_\varepsilon)\frac{\nabla u_\varepsilon}{(1+|u_\varepsilon|)}\right| dx &\leq C_g \int_\Omega \left| \frac{\nabla u_\varepsilon}{(1+|u_\varepsilon|)^{\frac{1}{p_l}}} \frac{u_\varepsilon^{s-\eta}}{(1+|u_\varepsilon|)^{1-\frac{1}{p_l}}}\right| \\
&\leq \frac{C_A}{2}\sum_{l=1}^N \int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx + C_1 \sum_{l=1}^N \int_\Omega \frac{|u_\varepsilon|^{\frac{(s-\eta)p_l}{p_l-1}}}{(1+|u_\varepsilon|)}\,dx \\
&\leq \frac{C_A}{2}\sum_{l=1}^N \int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx + C_1 \sum_{l=1}^N \int_\Omega |u_\varepsilon|^{\frac{(s-\eta)p_l}{p_l-1}}\,dx \\
&\leq \frac{C_A}{2}\sum_{l=1}^N \int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx + C_1 \sum_{l=1}^N \int_\Omega (1+|u_\varepsilon|)^{\frac{(s-\eta)p_l}{p_l-1}}\,dx \\
&\leq \frac{C_A}{2}\sum_{l=1}^N \int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx + C_2 \int_\Omega (1+|u_\varepsilon|)\,dx
\end{aligned}
\tag{5.4}
$$

Following [4], we shall modify the proofs of Proposition 4.1 and Lemma 2.2. Inserting the test function $\varphi = \log(1+|u_\varepsilon|)\mathrm{sign}(u_\varepsilon)$ into the weak formulation for u_ε and using (5.4), we find after some work the following a priori estimate:

$$
\begin{aligned}
\frac{C_A}{2}\sum_{l=1}^N \int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx &\leq C_3\int_\Omega (1+|u_\varepsilon|)\,dx + C_4\int_\Omega f_\varepsilon \log(1+|u_\varepsilon|)\,dx \\
&\leq C_4 \int_\Omega |f_\varepsilon|\log(1+|f_\varepsilon|)\,dx + (C_3+C_4)\int_\Omega (1+|u_\varepsilon|)\,dx \\
&\leq C_5 + C_6\int_\Omega (1+|u_\varepsilon|)\,dx,
\end{aligned}
\tag{5.5}
$$

where we have used the well-known inequality $xy \leq x\log(1+x) + \exp(y)$ for $x,y\geq 0$.

To turn (5.5) into an $L^{q_l}(\Omega)$ estimate on $\frac{\partial u_\varepsilon}{\partial x_l}$, we proceed as in (2.12). As in the proof of Proposition 4.1, one can prove that u_ε is uniformly (in ε) bounded in $L^s(\Omega)$ and thus $L^1(\Omega)$. By Hölder's inequality and then (5.5),

$$
\begin{aligned}
\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{q_l} dx &\leq \left(\int_\Omega \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)}\,dx\right)^{\frac{q_l}{p_l}} \left(\int_\Omega (1+|u_\varepsilon|)^{\frac{q_l}{p_l-q_l}}\,dx\right)^{\frac{p_l-q_l}{p_l}} \\
&\leq C_7 \left(\int_\Omega (1+|u_\varepsilon|)^{\frac{q_l}{p_l-q_l}}\,dx\right)^{\frac{p_l-q_l}{p_l}}.
\end{aligned}
\tag{5.6}
$$

Inserting (5.6) into (2.11) and keeping in mind that $\dfrac{q_l}{p_l-q_l} < \overline{q}^\star$, we find

$$
\begin{aligned}
\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}^\star}} &\leq C_8 + C_9 \prod_{l=1}^N \left(\int_\Omega (1+|u_\varepsilon|)^{\frac{q_l}{p_l-q_l}}\,dx\right)^{\frac{p_l-q_l}{q_l p_l N}} \\
&\leq C_{10} + C_{11}\left(\int_\Omega |u_\varepsilon|^{\overline{q}^\star}\,dx\right)^{\frac{1}{\overline{q}}-\frac{1}{\overline{p}}},
\end{aligned}
$$

and then we obtain (5.3) as in the proof of Lemma 2.2. □

6. Parabolic Case

We consider nonlinear anisotropic parabolic equations of the form

$$
\begin{cases} u_t - \mathrm{div} A(t,x,\nabla u) - \mathrm{div} g(t,x,u) + h(t,x,u) = f(t,x) & (t,x) \in (0,T)\times\mathbb{R}^N, \\ u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \end{cases} \tag{6.1}
$$

where $T>0$ is a fixed number. The vector field $A : (0,T)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ has components $a_l : (0,T)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$, $l=1,\dots,N$, and we assume that there exist two constants C_A and C_A' such that for all $\xi_1,\xi_2\in\mathbb{R}^N$ and for a.e. (t,x)

$$
A(t,x,\xi)\cdot\xi \ge C_A \sum_{l=1}^{N} |\xi|^{p_l}, \tag{6.2}
$$

$$
|a_l(t,x,\xi)| \le C_A' \left(1 + \sum_{\ell=1}^{N} |\xi_\ell|^{p_\ell - 1}\right), \qquad l=1,\dots,N, \tag{6.3}
$$

$$
\left[A(t,x,\xi_1) - A(t,x,\xi_2)\right]\left[\xi_1 - \xi_2\right] > 0, \qquad \xi_1 \ne \xi_2. \tag{6.4}
$$

We assume that the advection field $g : (0,T)\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}^N$ has continuous components $g_l : (0,T)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, $l=1,\dots,N$, and satisfies the following conditions:

$$
|g(t,x,\sigma)| \le C_g |\sigma|^{s-\eta}, \qquad \text{for a.e. } (t,x)\in(0,T)\times\mathbb{R}^N \text{ and for all } \sigma\in\mathbb{R}. \tag{6.5}
$$

$$
|\mathrm{div}_x g(t,x,\sigma)| \le C_g' |\sigma|^{s-\eta}, \qquad \text{for a.e. } (t,x)\in(0,T)\times\mathbb{R}^N \text{ and for all } \sigma\in\mathbb{R}, \tag{6.6}
$$

for some constants C_g, C_g' and some $\eta\in(1,s)$.

The function $h : (0,T)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is assumed to be measurable in $(t,x)\in(0,T)\times\mathbb{R}^N$ for all $\sigma\in\mathbb{R}$ and continuous in $\sigma\in\mathbb{R}$ for a.e. $(t,x)\in(0,T)\times\mathbb{R}^N$. Furthermore,

$$
h(t,x,\sigma)\sigma \ge 0, \qquad \text{for all } \sigma\in\mathbb{R} \text{ and a.e. } (t,x)\in(0,T)\times\mathbb{R}^N, \tag{6.7}
$$

$$
\sup\{|h(t,x,\sigma)| : |\sigma|\le\tau\} \in L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N)), \qquad \forall \tau\in\mathbb{R}. \tag{6.8}
$$

Finally, there should exist $s > p_l$, $l=1,\dots,N$, such that

$$
h(t,x,\sigma)\mathrm{sign}(\sigma) \ge |\sigma|^s, \qquad \text{for all } \sigma\in\mathbb{R} \text{ and a.e. } (t,x)\in(0,T)\times\mathbb{R}^N. \tag{6.9}
$$

The data f, u_0 are assumed to satisfy

$$
f\in L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N)), \qquad u_0\in L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{6.10}
$$

We assume that the exponents $p_1,\dots,p_N$ and s satisfy the following conditions:

$$
\begin{cases} \overline{p} < N + \dfrac{N}{N+1}, \qquad \dfrac{1}{\overline{p}} = \dfrac{1}{N}\displaystyle\sum_{l=1}^{N}\dfrac{1}{p_l}, \\ 2 - \dfrac{1}{N+1} < p_l < \dfrac{\overline{p}(N+1)}{N}, \qquad l=1,\dots,N, \\ s > p_l, \qquad l=1,\dots,N. \end{cases} \tag{6.11}
$$

We seek solutions to (6.1) in the following sense:

Definition 6.1. A distributional solution of (6.1) is a function

$$
u\in L^1\left(0,T;W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)\right)\cap L^s\left(0,T,L^s_{\mathrm{loc}}(\mathbb{R}^N)\right), \qquad A(t,x,\nabla u)\in\left(L^1\left(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N)\right)\right)^N,
$$

that satisfies

$$
\begin{aligned} &-\int_0^T\int_{\mathbb{R}^N} u\varphi_t\,dx\,dt + \int_0^T\int_{\mathbb{R}^N} \left(A(t,x,\nabla u) + g(t,x,u)\right)\cdot\nabla\varphi\,dx\,dt \\ &\quad + \int_0^T\int_{\mathbb{R}^N} h(t,x,u)\varphi\,dx\,dt = \int_0^T\int_{\mathbb{R}^N} f\varphi\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\varphi(0,x)\,dx, \end{aligned} \tag{6.12}
$$

for all $\varphi\in C_0^1([0,T)\times\mathbb{R}^N)$.

Our main existence result for (6.1) is stated the following theorem:

Theorem 6.1. *Assume* (6.2)-(6.10) *hold and that the corresponding exponents* $p_1, \dots, p_N$ *and* s *are restricted as in* (6.11). *Then* (6.1) *has at least one distributional solution* u. *If* $f, u_0 \geq 0$, *then* $u \geq 0$. *Moreover,* u *possesses the regularity*

$$u \in \bigcap_{l=1}^{N} L^{q_l}\left(0, T, W^{1,q_l}_{\mathrm{loc}}(\mathbb{R}^N)\right), \qquad 1 \leq q_l < \frac{p_l}{\overline{p}}\left(\overline{p} - \frac{N}{N+1}\right). \tag{6.13}$$

Finally, if $f, u_0 \in L^1(\mathbb{R}^N)$ *and* $\overline{p} > N$, *then* $u \in L^\infty_{\mathrm{loc}}((0,T) \times \mathbb{R}^N)$.

Proof. The proof is similar to the proof of Theorem 3.1, so we just sketch it. Let $\{f_\varepsilon\}_{0<\varepsilon\leq 1}$ and $\{u_{0,\varepsilon}\}_{0<\varepsilon\leq 1}$ be sequences functions satisfying

$$\begin{cases} f_\varepsilon \in C^\infty_c([0,T] \times \mathbb{R}^N) \quad \text{and} \quad u_{0,\varepsilon} \in C^\infty_c(\mathbb{R}^N); \\ |f_\varepsilon| \leq \dfrac{1}{\varepsilon}, \quad |f_\varepsilon| \leq |f|, \quad f_\varepsilon \to f \quad \text{in } L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \text{ as } \varepsilon \to 0; \\ |u_{0,\varepsilon}| \leq \dfrac{1}{\varepsilon}, \quad |u_{0,\varepsilon}| \leq |u_0|, \quad u_{0,\varepsilon} \to u_0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } \varepsilon \to 0; \end{cases} \tag{6.14}$$

Set $R = \dfrac{1}{\varepsilon}$. Then, classical results, see, e.g, [12, 9], provide the existence of a sequence of functions

$$\begin{cases} u_\varepsilon \in \displaystyle\bigcap_{l=1}^{N} L^{p_l}(0,T; W^{1,p_l}_0(B_R)) \cap L^s((0,T) \times B_R) \cap C([0,T]; L^2(B_R)), \\ \partial_t u_\varepsilon \in \displaystyle\sum_{l=1}^{N} L^{p_l'}\left(0,T; \left(W^{1,p_l}_0(B_R)\right)'\right), \end{cases}$$

each of them satisfying the weak formulation

$$\begin{aligned} &\int_0^T \langle \partial_t u_\varepsilon, \varphi\rangle \, dt + \int_0^T \int_{B_R} (A(t,x,\nabla u_\varepsilon) + g(t,x,u_\varepsilon)) \cdot \nabla\varphi \, dx\, dt \\ &\qquad + \int_0^T \int_\Omega h(t,x,u_\varepsilon)\varphi \, dx\, dt = \int_0^T \int_\Omega f_\varepsilon \varphi \, dx\, dt, \end{aligned} \tag{6.15}$$

for all $\varphi \in \bigcap_{l=1}^{N} L^{p_l}\left(0,T; W^{1,p_l}_0(B_R)\right) \cap L^\infty((0,T) \times B_R)$. Moreover, the maximum principle holds, so that $u_{0,\varepsilon} \geq 0$ and $f_\varepsilon \geq 0$ imply $u_\varepsilon \geq 0$.

We introduce the function

$$\psi_\gamma(\sigma) = \int_0^\sigma \varphi_\gamma(s)\, ds, \qquad \text{where } \varphi_\gamma \text{ is defined in (4.7).} \tag{6.16}$$

As in the proof of Proposition (4.1), we take $\varphi = \varphi_\gamma(u_\varepsilon)\theta^\alpha$ as a test function in (6.15) and find

$$\begin{aligned} &\int_{B_R} \psi_\gamma(u_\varepsilon(x,T))\theta^\alpha \, dx + \int_0^T \int_{B_R} A(t,x,\nabla u_\varepsilon) \cdot \nabla u_\varepsilon \varphi_\gamma'(u_\varepsilon)\theta^\alpha \, dx\, dt \\ &\quad + \int_0^T \int_{B_R} g(t,x,u_\varepsilon) \cdot \nabla u_\varepsilon \varphi_\gamma'(u_\varepsilon)\theta^\alpha \, dx\, dt + \int_0^T \int_{B_R} h(t,x,u_\varepsilon)\varphi_\gamma(u_\varepsilon)\theta \, dx\, dt \\ &\quad + \alpha \int_0^T \int_{B_R} A(t,x,\nabla u_\varepsilon) \cdot \nabla\theta \varphi_\gamma(u_\varepsilon)\theta^{\alpha-1} \, dx\, dt \\ &\quad + \alpha \int_0^T \int_{B_R} g(t,x,u_\varepsilon) \cdot \nabla\theta \varphi_\gamma(u_\varepsilon)\theta^{\alpha-1} \, dx\, dt \\ &\quad = \int_{B_R} \psi_\gamma(u_{0,\varepsilon})\theta^\alpha \, dx + \int_0^T \int_{B_R} f_\varepsilon \varphi_\gamma(u_\varepsilon)\theta^\alpha \, dx\, dt. \end{aligned}$$

We choose γ and α according to (4.9).

Proceeding as in the proof of Proposition 4.1 up to (4.16), we find eventually that

$$\begin{aligned}
&\int_{B_R} \psi_\gamma(u_\varepsilon(x,T))\theta(x)^\alpha\,dx + \frac{C_A}{2}\int_0^T\int_{B_R}\sum_{l=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}\varphi'_\gamma(u_\varepsilon)\theta^\alpha\,dx\,dt\\
&\qquad + \frac{1}{2}\int_0^T\int_{B_R} h(t,x,u_\varepsilon)\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx\,dt \qquad (6.17)\\
&\qquad\qquad \le \int_{B_{2\rho}}\psi_\gamma(u_0(x))\,dx + \int_0^T\int_{B_{2\rho}}|f|\,dx\,dt + C_1 T\mathrm{meas}(B_{2\rho}),
\end{aligned}$$

which in turn implies the existence of a constant C_2, independent of ε as long as $R = \frac{1}{\varepsilon} > 2\rho$ for a given ρ, such that

$$\sum_{l=1}^N \int_0^T\int_{B_\rho} \frac{\left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p_l}}{(1+|u_\varepsilon|)^\gamma}\,dx\,dt \le C_2 \tag{6.18}$$

and

$$\int_0^T\int_{B_\rho} |u_\varepsilon|^s\,dx\,dt \le C_2. \tag{6.19}$$

Note that from definition of ψ_γ and estimate (6.17), we deduce also that

$$\sup_{t\in(0,T)}\int_{B_\rho}|u_\varepsilon|\,dx \le C_2. \tag{6.20}$$

In view of (6.18), (6.19), (6.20), we can carry out the "interpolation" step as in [2, 11] and obtain

$$\left\|\frac{\partial u_\varepsilon}{\partial x_l}\right\|_{L^{q_l}((0,T)\times B_\rho)} \le C_3, \qquad l = 1,\dots,N, \tag{6.21}$$

and

$$\|u_\varepsilon\|_{L^{\overline{q}}((0,T)\times B_\rho))} \le C_4, \tag{6.22}$$

for every q_l satisfying the condition in (6.13).

Let $q_0 = \min_{1\le l\le N} q_l$. Given any $\rho > 0$,

$$u_\varepsilon \text{ is uniformly (in } \varepsilon\text{) bounded in } L^{q_0}\left(0,T;W^{1,q_0}(B_\rho)\right) \text{ as long as } \tfrac{1}{\varepsilon} > 2\rho.$$

This is a consequence of (6.21) and (6.22). This implies that

$$\partial_t u_\varepsilon \text{ is uniformly (in } \varepsilon\text{) bounded in } L^1\left(0,T;\left(W^{1,q_0}(B_\rho)\right)'\right) + L^1\left(0,T;L^1(B_\rho)\right).$$

We can therefore assume that as $\varepsilon \to 0$ (see, e.g., [15, Corollary 4])

$$u_\varepsilon \to u \quad \text{strongly in } L^{q_0}((0,T)\times B_\rho) \text{ for any } \rho \text{ and a.e. in } (0,T)\times\mathbb{R}^N,$$

and $h(t,x,u_\varepsilon) \to h(t,x,u)$ a.e. in $(0,T)\times\mathbb{R}^N$.

We prove the strong convergence in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ of the sequences $(h(t,x,u_\varepsilon))_{0<\varepsilon\le 1}$, $(g(t,x,u_\varepsilon))_{0<\varepsilon\le 1}$, and $(A(t,x,\nabla u_\varepsilon))_{0<\varepsilon\le 1}$ as in the proofs of Propositions 4.2 and 4.3. Hence we conclude that the limit function u is a distribution solution of (1.1) possessing the regularity stated in (6.13). □

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